Non-Singular Configuration Analyses of Redundant Manipulators for Optimizing Avoidance Manipulability

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***Abstract*—This paper is concerned with a new concept of avoidance manipulability inspired from manipulability. The manipulability represents the ability to generate velocity at the tip of each link without any designated hand task. The avoidance manipulability represents the shape-changeability (avoidance ability) of each intermediate link when a prior hand task is given. The intermediate links represents all comprising links of robot except top link with end-effector. The avoidance matrices, $^1\boldsymbol{M}_i$ ($i = 1, \cdots, n-1$) corresponding to all i-th intermediate links, are used for analyzing avoidance manipulability, resulting in that $rank(^1\boldsymbol{M}_i)$ declares the shape-changeable space expansion and singular values of $^1\boldsymbol{M}_i$ indicates the avoidance ability of i-th link. In this research, what assumption can guarantee mathematically the configurations with maximum $rank(^1\boldsymbol{M}_i)$ is our main concern for maximizing shape-changeability to prepare effectively dynamic change of environment or sudden appearance of obstacles. Then we proved that our "Non-singular Configuration Assumption" can guarantee the maximum rank of $^1\boldsymbol{M}_i$ through detailed decomposition analyses of $^1\boldsymbol{M}_i$.**

I. INTRODUCTION

Kinematically redundant manipulators have more DoF than necessary for accomplishing a given hand task. Nowadays, redundant manipulators are used for various kinds of tasks such as welding, sealing, grinding and contact tasks. Many kinematic researches are usually used to solve the problem of motion and obstacle avoidance of redundant manipulators discussing how to use the redundancy. Up to now, a variety of indices have been proposed for evaluation of the performance of robot manipulators. The manipulability ellipsoid [1] was presented to evaluate the static performance of a robot manipulator as an index evaluating the manipulator's shape on the view point of how much the hand velocity can be generated by normalized joint velocity. Further, [2] formulated the relation of the redundancy and the priority order of multiple tasks. [3] proposed a control method of the redundancy based on priority order of tasks, and pointed out the effectiveness by actual experiments. A method that uses perturbation of a dumping element for avoiding obstacles along with singular configuration, and a regulation method of dumping element were discussed in [4]. The manipulability measure was addressed for cooperative arms [5] and for dexterous hands [6] and was used in real-time control [7]. In

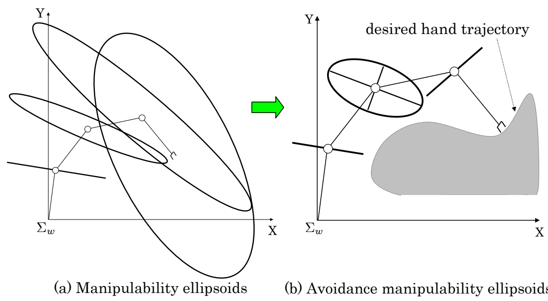
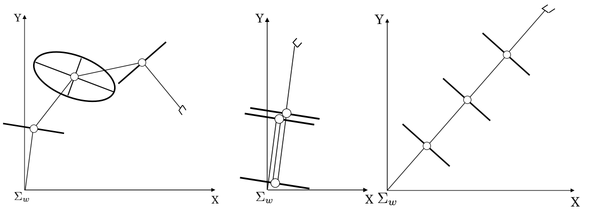


Fig. 1. Manipulability ellipsoids and Avoidance manipulability ellipsoids

addition, the manipulating force ellipsoid [8] was presented to evaluate the static torque-force transmission from the joints to the end-effector, while the dynamic manipulability ellipsoid [9] was presented as an index of the dynamic performance of a robot manipulator. Recent years, combining the dynamic manipulability ellipsoid with the manipulability force ellipsoid, the inertia matching ellipsoid [10] was proposed to characterize the dynamic torque-force transmission efficiency.

The researches mentioned above were an argument in a condition that an assumption guarantees the possibility that multiple avoiding motions could be realized. They did not consider how much residue redundant freedoms are remained at the links required to avoid obstacle. However, in an on-line system with dynamic environment, when a moving obstacle appears suddenly near the manipulator, it requires the manipulator to possess the ability to avoid this moving obstacle by changing its shape, which is so-called "Avoidance Manipulability". In this background, as shown in Fig.1, we had presented the avoidance manipulability ellipsoid concept as an index evaluating shape-changeability of the manipulator [11], which is inspired from the manipulability concept [1]. In fact, the avoidance matrix ($^1\boldsymbol{M}_i$), which is important to analyze avoidance manipulability, had initially been defined and used for controling the redundant manipulator's configuration based on prioritized multiple tasks [12]. However, the proposed controller can not decouple the interacting motions of multiple tasks even though the redundant degree be much higher than the required motion degree of the multiple tasks, stemming from incompleted definition of Jacobian matrix concerning the motion of what number of links the matrix describes. Contrasting [12] with our definition of Jacobian matrix, the detailed difference and explanation are shown in sections III.



(a) Non-singular configuration (b) Singular configurations

Fig. 2. Non-singular configuration and Singular configurations

Avoidance manipulability of the manipulator is evaluated in the residue redundant surplus space depending on the avoidance ability in each possible dimension of intermediate links, where intermediate link means all links except tip link with hand. In other words, maximum avoidance space expasion ($rank(^1\boldsymbol{M}_i)$) and maximum avoidance ability in possible avoidance dimension (singular values of $^1\boldsymbol{M}_i$) will determine the avoidance manipulability. In this paper, what assumption guarantees mathematically the configurations with $rank(^1\boldsymbol{M}_i)$ as shown in Fig.2(a) is our main problem, that is, aiming at avoiding the singular configurations as shown in Fig.2(b). We discuss this assumption named as "Non-singular Configuration Assumption", which can assure the rank of $^1\boldsymbol{M}_i$ to be maximized, through analysis and proof by decomposing $^1\boldsymbol{M}_i$ into singular components. Maximization of $rank(^1\boldsymbol{M}_i)$ of intermediate links is the essential requirement for configuration optimization of manipulator with high avoidance manipulability. And it is the first step of design for an on-line control system of a redundant manipulator with high shape-changeability through avoidance manipulability.

II. REDUNDANT MANIPULATOR'S KINEMATICS

A. Analysis in Position Space

Representing the position vector of i-th link by $\boldsymbol{r}_{p,i} \in R^{m_p}$ ($i = 1, 2, \cdots, n$). m_p denotes the position dimension number of working space ($1 \leq m_p \leq 3$), n denotes the number of the manipulator's links and $m_p < n$ because of redundancy. $\boldsymbol{r}_{p,i}$ is given as a function of $\boldsymbol{q}_i$ and defined as

$$\boldsymbol{r}_{p,i} = \boldsymbol{r}_{p,i}(\boldsymbol{q}_i) = [r_{1,i}(\boldsymbol{q}_i), \cdots, r_{m_p,i}(\boldsymbol{q}_i)]^T \tag{1}$$

In (1), $\boldsymbol{q}_i$ with n elements is defined as

$$\boldsymbol{q}_i = [q_1, \cdots, q_i, 0, \cdots, 0]^T, \quad (i = 1, 2, \cdots, n) \tag{2}$$

In addition, according to Fig.3, $\boldsymbol{r}_{p,i}(\boldsymbol{q}_i)$ can be denoted as

$$\boldsymbol{r}_{p,i}(\boldsymbol{q}_i) \quad = \quad \sum_{j=1}^{i} \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j) \tag{3}$$

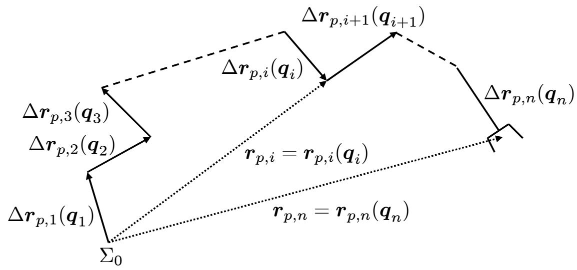


Fig. 3. Structure 1 of n-link redundant manipulator

By differentiating $\boldsymbol{r}_{p,i}(\boldsymbol{q}_i)$ in (3) with time, we can obtain

$$\begin{aligned}\dot{\boldsymbol{r}}_{p,i}(\boldsymbol{q}_i) &= \frac{\partial \boldsymbol{r}_{p,i}(\boldsymbol{q}_i)}{\partial \boldsymbol{q}_n^T}\dot{\boldsymbol{q}}_n \\ &= \frac{\partial \Delta\boldsymbol{r}_{p,1}(\boldsymbol{q}_1)}{\partial \boldsymbol{q}_1^T}\dot{\boldsymbol{q}}_n + \cdots + \frac{\partial \Delta\boldsymbol{r}_{p,i}(\boldsymbol{q}_i)}{\partial \boldsymbol{q}_i^T}\dot{\boldsymbol{q}}_n \\ &= \boldsymbol{J}_{p,i}\dot{\boldsymbol{q}}_n \end{aligned} \tag{4}$$

Then, we can obtain the position Jacobian matrix $\boldsymbol{J}_{p,i}$ ($i = 1, 2, \cdots, n$) in (4) as follows:

$$\begin{aligned}\boldsymbol{J}_{p,i} &= [\underbrace{\sum_{j=1}^{i}\frac{\partial \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_1}, \; \cdots, \; \frac{\partial \Delta\boldsymbol{r}_{p,i}(\boldsymbol{q}_i)}{\partial q_i}}_{i}, \; \underbrace{\boldsymbol{0}}_{n-i}]\}m_p \\ &= [\tilde{\boldsymbol{j}}_{p,i,1}, \; \cdots, \; \tilde{\boldsymbol{j}}_{p,i,i}, \; \boldsymbol{0}] = [\tilde{\boldsymbol{J}}_{p,i}, \; \boldsymbol{0}] \end{aligned} \tag{5}$$

If we redefine $\Delta\boldsymbol{J}_{p,j}$ as

$$\begin{aligned}\Delta\boldsymbol{J}_{p,j} &= [\underbrace{\frac{\partial \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_1}, \; \cdots, \; \frac{\partial \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_j}}_{j}, \; \underbrace{\boldsymbol{0}}_{n-j}]\}m_p \\ &= [\Delta\tilde{\boldsymbol{j}}_{p,j,1}, \; \cdots, \; \Delta\tilde{\boldsymbol{j}}_{p,j,j}, \; \boldsymbol{0}] = [\Delta\tilde{\boldsymbol{J}}_{p,j}, \; \boldsymbol{0}] \end{aligned} \tag{6}$$

$\boldsymbol{J}_{p,i}$ ($i = 1, 2, \cdots, n$) can be denoted as

$$\boldsymbol{J}_{p,i} = \sum_{j=1}^{i}\Delta\boldsymbol{J}_{p,j} \tag{7}$$

In this way, $\boldsymbol{J}_{p,n}$ can be denoted as

$$\boldsymbol{J}_{p,n} \quad = \quad \sum_{j=1}^{n}\Delta\boldsymbol{J}_{p,j} = \boldsymbol{J}_{p,i} + \sum_{j=i+1}^{n}\Delta\boldsymbol{J}_{p,j} \tag{8}$$

In addition, referring to Fig.4, we know

$$^0\boldsymbol{p}_{i+1,k} = \sum_{j=k}^{i}\Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j) = \sum_{j=k}^{i}{}^0\boldsymbol{R}_j{}^j\hat{\boldsymbol{p}}_{j+1} \tag{9}$$

In (9), $^0\boldsymbol{R}_j$ is rotation matrix denoting the relation between Σ_0 and Σ_j, $^j\hat{\boldsymbol{p}}_{j+1}$ is the constant denoting position vector from the origin of Σ_j to the one of Σ_{j+1} with respect to Σ_j. Then, we can obtain

$$\Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j) = {}^0\boldsymbol{R}_j{}^j\hat{\boldsymbol{p}}_{j+1} \tag{10}$$

Then, by differentiating $\Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)$ with time, we can obtain

$$\frac{d\Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{dt} = \frac{\partial \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \Delta\boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_j}\dot{q}_j \tag{11}$$

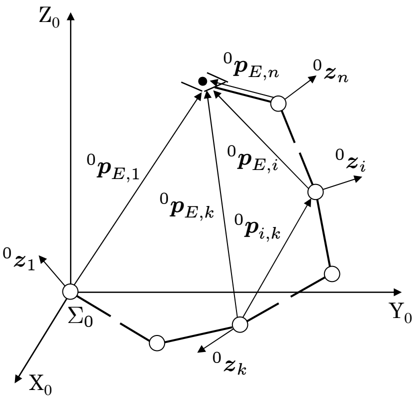


Fig. 4. Structure 2 of n-link redundant manipulator

Given the i-th joint be rotational,

$$\frac{\partial \Delta \boldsymbol{r}_{p,j}(\boldsymbol{q}_j)}{\partial q_i} = {}^0\boldsymbol{z}_i \times ({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1}) \quad (12)$$

since,

$$\begin{aligned} \frac{d({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1})}{dt} &= \frac{d{}^0\boldsymbol{R}_j}{dt} {}^j\hat{\boldsymbol{p}}_{j+1} = (\sum_{i=1}^{j} {}^0\boldsymbol{z}_i \dot{q}_i) \times {}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1} \\ &= {}^0\boldsymbol{z}_1 \times ({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1})\dot{q}_1 + \cdots \\ &+ {}^0\boldsymbol{z}_j \times ({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1})\dot{q}_j \end{aligned} \quad (13)$$

where, ${}^0\boldsymbol{z}_i$ denotes the unit vector of axis direction of i-th joint defined as

$${}^0\boldsymbol{z}_i = {}^0\boldsymbol{R}_i \boldsymbol{e}_z \quad (14)$$

and in (14), $\boldsymbol{e}_z = [0, 0, 1]^T$ providing z-axes of all link coordinates represent rotational axes. Substituting (12) into (6), we can obtain

$$\Delta \boldsymbol{J}_{p,j} = [\underbrace{{}^0\boldsymbol{z}_1 \times ({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1}), \cdots, {}^0\boldsymbol{z}_j \times ({}^0\boldsymbol{R}_j {}^j\hat{\boldsymbol{p}}_{j+1})}_{j}, \underbrace{\boldsymbol{0}}_{n-j}]\}m_p \quad (15)$$

Then, according to (7) and (9), we can obtain

$$\boldsymbol{J}_{p,i} = [\underbrace{{}^0\boldsymbol{z}_1 \times {}^0\boldsymbol{p}_{i+1,1}, \cdots, {}^0\boldsymbol{z}_i \times {}^0\boldsymbol{p}_{i+1,i}}_{i}, \underbrace{\boldsymbol{0}}_{n-i}]\}m_p \quad (16)$$

If the k-th joint is prismatic, the translational direction is represented by ${}^0\boldsymbol{z}_k$ ($1 \le k \le i$), then $\boldsymbol{J}_{p,i}$ is denoted as

$$\boldsymbol{J}_{p,i} = [\underbrace{{}^0\boldsymbol{z}_1 \times {}^0\boldsymbol{p}_{i+1,1}, \cdots}_{k-1}, \underbrace{{}^0\boldsymbol{z}_k}_{1}, \underbrace{\cdots, {}^0\boldsymbol{z}_i \times {}^0\boldsymbol{p}_{i+1,i}}_{i-k}, \underbrace{\boldsymbol{0}}_{n-i}]\}m_p \quad (17)$$

B. Analysis in Orientation Space

Representing the orientation vector of each link by $\boldsymbol{r}_{o,i} \in R^{m_o}$. Here, m_o denotes the orientation dimension number of working space ($1 \le m_o \le 3$). If $\boldsymbol{r}_{o,i}$ is represented by a rather common definition of "Euler angles" $(\phi_i, \theta_i, \psi_i)$, and it is given as a function of $\boldsymbol{q}_i$ and defined as

$$\boldsymbol{r}_{o,i} = \boldsymbol{r}_{o,i}(\boldsymbol{q}_i) = [\phi_i(\boldsymbol{q}_i), \theta_i(\boldsymbol{q}_i), \psi_i(\boldsymbol{q}_i)]^T \quad (18)$$

By differentiating $\boldsymbol{r}_{o,i}$ in (18) with time, we can obtain

$$\dot{\boldsymbol{r}}_{o,i}(\boldsymbol{q}_i) = \frac{\partial \boldsymbol{r}_{o,i}(\boldsymbol{q}_i)}{\partial \boldsymbol{q}_n^T} \dot{\boldsymbol{q}}_n = \bar{\boldsymbol{J}}_{o,i} \dot{\boldsymbol{q}}_n \quad (19)$$

In addition, the relation between angular velocity vector $\boldsymbol{\omega}_i$ and $\dot{\boldsymbol{r}}_{o,i}(\boldsymbol{q}_i)$ is

$$\begin{aligned} \boldsymbol{\omega}_i &= \begin{bmatrix} 0 & -sin\phi_i & cos\phi_i sin\theta_i \\ 0 & cos\phi_i & sin\phi_i sin\theta_i \\ 1 & 0 & cos\theta_i \end{bmatrix} \dot{\boldsymbol{r}}_{o,i}(\boldsymbol{q}_i) \\ &= \begin{bmatrix} 0 & -sin\phi_i & cos\phi_i sin\theta_i \\ 0 & cos\phi_i & sin\phi_i sin\theta_i \\ 1 & 0 & cos\theta_i \end{bmatrix} \\ &\quad \begin{bmatrix} \frac{\partial \phi_i(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial \phi_i(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \\ \frac{\partial \theta_i(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial \theta_i(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \\ \frac{\partial \psi_i(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial \psi_i(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \end{bmatrix} \dot{\boldsymbol{q}}_n \\ &= \boldsymbol{J}_{o,i} \dot{\boldsymbol{q}}_n \end{aligned} \quad (20)$$

In (20), providing z-axes of all links represent rotational axes, $\boldsymbol{J}_{o,i}$ is denoted as

$$\boldsymbol{J}_{o,i} = [\underbrace{{}^0\boldsymbol{z}_1, \cdots, {}^0\boldsymbol{z}_i}_{i}, \underbrace{\boldsymbol{0}}_{n-i}]\}m_o \quad (21)$$

If the k-th joint is prismatic ($1 \le k \le i$), $\boldsymbol{J}_{o,i}$ is denoted as

$$\boldsymbol{J}_{o,i} = [\underbrace{{}^0\boldsymbol{z}_1, \cdots, {}^0\boldsymbol{z}_{k-1}}_{k-1}, \underbrace{\boldsymbol{0}}_{1}, \underbrace{{}^0\boldsymbol{z}_{k+1}, \cdots, {}^0\boldsymbol{z}_i}_{i-k}, \underbrace{\boldsymbol{0}}_{n-i}]\}m_o \quad (22)$$

Being similar with (7), $\boldsymbol{J}_{o,i}$ can be denoted as

$$\boldsymbol{J}_{o,i} = \sum_{j=1}^{i} \Delta \boldsymbol{J}_{o,j} \quad (23)$$

C. Analysis in Both Position and Orientation Spaces

According to above analyses of Jacobian matrices in position space ($1 \le m_p \le 3$) and orientation space ($1 \le m_o \le 3$) respectively. Firstly, when $m_p = 3$, $\boldsymbol{r}_{p,i}(\boldsymbol{q}_i) = [x(\boldsymbol{q}_i), y(\boldsymbol{q}_i), z(\boldsymbol{q}_i)]^T$. Then, from (5) we can define a general positon Jacobian matrix ($3 \times n$) as

$$\boldsymbol{J}_{p,i}^{m_p=3} = \begin{bmatrix} \frac{\partial x(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial x(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \\ \frac{\partial y(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial y(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \\ \frac{\partial z(\boldsymbol{q}_i)}{\partial q_1} & \cdots & \frac{\partial z(\boldsymbol{q}_i)}{\partial q_i} & \boldsymbol{0} \end{bmatrix} \quad (24)$$

Next, when $m_0 = 3$ and $\boldsymbol{r}_{o,i}(\boldsymbol{q}_i)$ is defined as (18) in orientation space. Then, we can define a general orientation Jacobian matrix $\boldsymbol{J}_{o,i}^{m_o=3}$ ($3 \times n$), which is just $\boldsymbol{J}_{o,i}$ analysed from (20) to (23) in orientation space.

Finally, if a general Jacobian matrix ($6 \times n$) in the maximum space of $m = m_p + m_o = 6$ such as $\boldsymbol{r}_i(\boldsymbol{q}_i) = [x(\boldsymbol{q}_i), y(\boldsymbol{q}_i), z(\boldsymbol{q}_i), \phi_i(\boldsymbol{q}_i), \theta_i(\boldsymbol{q}_i), \psi_i(\boldsymbol{q}_i)]^T$ is defined as

$$\boldsymbol{J}_i^{m=6} = \begin{bmatrix} \boldsymbol{J}_{p,i}^{m_p=3} \\ \boldsymbol{J}_{o,i}^{m_o=3} \end{bmatrix} \quad (25)$$

In this way, according to $\boldsymbol{J}_i^{m=6}$, we can define any Jacobian matrix in any kind of space as

$$\begin{aligned}\boldsymbol{J}_i &= \boldsymbol{U}^m \boldsymbol{J}_i^{m=6} \\ &= [\underbrace{\tilde{\boldsymbol{j}}_{i,1},\ \cdots,\ \tilde{\boldsymbol{j}}_{i,i}}_{i}, \underbrace{\boldsymbol{0}}_{n-i}]\}m = [\tilde{\boldsymbol{J}}_i,\ \boldsymbol{0}]\end{aligned} \quad (26)$$

In (26) providing $i = n$, $\boldsymbol{U}^m$ is a $m\times6$ matrix and is used to select objective space used selectively for hand task. For example, when the objective space is given by 3-dimensional such as $\boldsymbol{r}_i(\boldsymbol{q}_i) = [x(\boldsymbol{q}_i), y(\boldsymbol{q}_i), \phi_i(\boldsymbol{q}_i)]^T$, $\boldsymbol{U}^m$ is 3×6 matrix as

$$\boldsymbol{U}^m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (27)$$

In addition, being similar with (7) and (23), we can define

$$\boldsymbol{J}_i = \sum_{j=1}^{i} \Delta \boldsymbol{J}_j \quad (28)$$

and

$$\Delta \boldsymbol{J}_j = \begin{bmatrix} \Delta \boldsymbol{J}_{p,j} \\ \Delta \boldsymbol{J}_{o,j} \end{bmatrix} \quad (29)$$

III. AVOIDANCE MANIPULABILITY

Here we assume that the desired trajectory ($\boldsymbol{r}_{nd}$) and the desired velocity of the manipulator's hand ($\dot{\boldsymbol{r}}_{nd}$) are given as primary task. Then, according to (30),

$$\dot{\boldsymbol{r}}_{nd} = \boldsymbol{J}_n \dot{\boldsymbol{q}}_n \quad (30)$$

we can obtain

$$\dot{\boldsymbol{q}}_n = \boldsymbol{J}_n^{+} \dot{\boldsymbol{r}}_{nd} + (\boldsymbol{I}_n - \boldsymbol{J}_n^{+} \boldsymbol{J}_n)\ {}^1\boldsymbol{l} \quad (31)$$

In (31), $\boldsymbol{J}_n$ is Jacobian matrix differentiated $\boldsymbol{r}_n$ by $\boldsymbol{q}_n$ ($\dot{\boldsymbol{r}}_n = \boldsymbol{J}_n \dot{\boldsymbol{q}}_n$), $\boldsymbol{J}_n^{+}$ is pseudo-inverse of $\boldsymbol{J}_n$, $\boldsymbol{I}_n$ is $n\times n$ unit matrix, and ${}^1\boldsymbol{l}$ is an arbitrary vector satisfying ${}^1\boldsymbol{l} \in R^n$. The left superscript "1" of ${}^1\boldsymbol{l}$ means the first avoidance sub-task executed by using redundant DoF. The following definitions about the left superscript "1" are also. By substituting (31) into ${}^1\dot{\boldsymbol{r}}_{di} = \boldsymbol{J}_i \dot{\boldsymbol{q}}_n$, the relation of ${}^1\dot{\boldsymbol{r}}_{di}$ and $\dot{\boldsymbol{r}}_{nd}$ is denoted as

$${}^1\dot{\boldsymbol{r}}_{di} = \boldsymbol{J}_i \boldsymbol{J}_n^{+} \dot{\boldsymbol{r}}_{nd} + \boldsymbol{J}_i(\boldsymbol{I}_n - \boldsymbol{J}_n^{+} \boldsymbol{J}_n)\ {}^1\boldsymbol{l} \quad (32)$$

Then, we define two variables shown as

$$\Delta {}^1\dot{\boldsymbol{r}}_{di} \stackrel{\triangle}{=} {}^1\dot{\boldsymbol{r}}_{di} - \boldsymbol{J}_i \boldsymbol{J}_n^{+} \dot{\boldsymbol{r}}_{nd} \quad (33)$$

and

$${}^1\boldsymbol{M}_i \stackrel{\triangle}{=} \boldsymbol{J}_i(\boldsymbol{I}_n - \boldsymbol{J}_n^{+} \boldsymbol{J}_n) \quad (34)$$

In (33), $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ is called by "the first avoidance velocity". In (34), ${}^1\boldsymbol{M}_i$ is a $m\times n$ matrix called by "the first avoidance matrix". Then, $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ can be rewritten as

$$\Delta {}^1\dot{\boldsymbol{r}}_{di} = {}^1\boldsymbol{M}_i {}^1\boldsymbol{l} \quad (35)$$

The relation between ${}^1\dot{\boldsymbol{r}}_{di}$ and $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ is shown in Fig.5. From (35), we can obtain ${}^1\boldsymbol{l}$ shown as

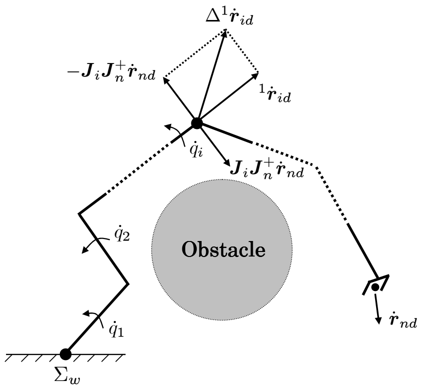


Fig. 5. Obstacle avoidance of intermediate links

$${}^1\boldsymbol{l} = {}^1\boldsymbol{M}_i{}^{+} \Delta {}^1\dot{\boldsymbol{r}}_{di} + (\boldsymbol{I}_n - {}^1\boldsymbol{M}_i^{+}\ {}^1\boldsymbol{M}_i)^2\boldsymbol{l} \quad (36)$$

In (36), ${}^1\boldsymbol{M}_i^{+}$ is pseudo-inverse of ${}^1\boldsymbol{M}_i$ and ${}^2\boldsymbol{l}$ is an arbitrary vector satisfying ${}^2\boldsymbol{l} \in R^n$. From (36), we can obtain

$$\|{}^1\boldsymbol{l}\|^2 \geq \Delta {}^1\dot{\boldsymbol{r}}_{di}{}^T ({}^1\boldsymbol{M}_i^{+})^T\ {}^1\boldsymbol{M}_i^{+} \Delta {}^1\dot{\boldsymbol{r}}_{di} \quad (37)$$

Assuming that ${}^1\boldsymbol{l}$ is restricted as $\|{}^1\boldsymbol{l}\| \leq 1$, then the extent where $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ can move is denoted as

$$\Delta {}^1\dot{\boldsymbol{r}}_{di}{}^T ({}^1\boldsymbol{M}_i^{+})^T\ {}^1\boldsymbol{M}_i^{+} \Delta {}^1\dot{\boldsymbol{r}}_{di} \leq 1 \quad (38)$$

When $rank({}^1\boldsymbol{M}_i) = m$, (38) represents that the first avoidance velocity $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ can be described by an ellipsoid expanded in m-dimensional space, which indicates $\Delta {}^1\dot{\boldsymbol{r}}_{di}$ can be freely realized in m-dimensional task space. The ellipsoid represented by (38) is named as the first complete avoidance manipulability ellipsoid. However, when $rank({}^1\boldsymbol{M}_i) = p < m$, the extent of the new first avoidance velocity $\Delta {}^1\dot{\boldsymbol{r}}_{di}^{*}$ is denoted as

$$\Delta {}^1\dot{\boldsymbol{r}}_{di}^{*}{}^T ({}^1\boldsymbol{M}_i^{+})^T\ {}^1\boldsymbol{M}_i^{+} \Delta {}^1\dot{\boldsymbol{r}}_{di}^{*} \leq 1 \quad (39)$$

This new first avoidance velocity $\Delta {}^1\dot{\boldsymbol{r}}_{di}^{*} = {}^1\boldsymbol{M}_i\ {}^1\boldsymbol{M}_i^{+} \Delta {}^1\dot{\boldsymbol{r}}_{di}$ can be described by an ellipsoid expanded in p-dimensional space. The ellipsoid represented by (39) is named as the first partial avoidance manipulability ellipsoid. Because $p < m$, the first partial avoidance manipulability ellipsoid can be thought as a segment of the first complete avoidance manipulability ellipsoid as first and third links shown in Fig.2(a). Thus $rank({}^1\boldsymbol{M}_i)$ determines the possible avoidance dimension of i-th link, therefore the condition to give $rank({}^1\boldsymbol{M}_i)$ maximum number is essential for configuration control and avoidance control to maximize the shape-changeability degree. Next we will propose an assumption and prove it guarantees the rank of ${}^1\boldsymbol{M}_i$.

IV. ANALYSIS OF $rank({}^1\boldsymbol{M}_i)$

A. Non-singular Configuration Assumption

"Non-singular Configuration Assumption" is

$$rank(\boldsymbol{J}_i{}^{\nu\rightarrow\nu+m-1}) = min\{i, m\}\ (1\leq\nu\leq i - m + 1) \quad (40)$$

In (40), $\boldsymbol{J}_i{}^{\nu\rightarrow\nu+m-1}$ indicates the matrices including the m column vectors sequentially chosen from the first i columns of $\boldsymbol{J}_i$ without the last $n-i$ zero columns. That is the m column vectors sequentially chosen from $\tilde{\boldsymbol{J}}_i$ as

$$rank(\tilde{\boldsymbol{J}}_i^{\nu\to\nu+m-1}) = min\{i, m\} \ (1\leq\nu\leq i-m+1) \quad (41)$$

In (41), for example, when $i = n$ and $\nu = n - m + 1$,

$$\tilde{\boldsymbol{J}}_n^{n-m+1\to n} = [\tilde{\boldsymbol{j}}_{n,n-m+1}, \cdots, \tilde{\boldsymbol{j}}_{n,n}] \quad (42)$$

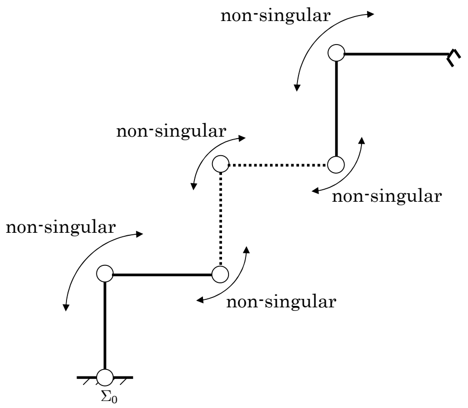


Fig. 6. Conceptual description of Non-singular configuration

"Non-singular Configuration Assumption" is a kind of mathematical denotation, which corresponds to the non-singular configuration described as Fig.6 in robot field.

B. Results

By (41), we will prove that we can obtain "$Results$" of $rank({}^1\boldsymbol{M}_i)$ $(i = 1, 2, \cdots, n-1)$ as follows:

1) Results in Both Position and Orientation Spaces $(\{m = m_p + m_o\}\cap\{2\leq m_p\leq 3\})$*:* When $n\geq 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < m) \\ m \ (m\leq i\leq n-m) \\ n-i\sim m \ (n-m < i\leq n-2) \\ 1\sim m-1 \ (i = n-1) \end{cases} \quad (43)$$

When $n < 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < n-m) \\ n-m \ (n-m\leq i\leq m) \\ n-i\sim n-m \ (m < i\leq n-1) \end{cases} \quad (44)$$

2) Results in Position Space $(\{m = m_p\}\cap\{2\leq m_p\leq 3\})$*:* When $n\geq 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < m) \\ m \ (m\leq i\leq n-m) \\ n-i\sim m \ (n-m < i\leq n-2) \\ 1\sim m-1 \ (i = n-1) \end{cases} \quad (45)$$

When $n < 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < n-m) \\ n-m \ (n-m\leq i\leq m) \\ n-i\sim n-m \ (m < i\leq n-1) \end{cases} \quad (46)$$

3) Other Results $(\{m = m_p = 1\}\cup\{m = m_o\}\cup\{m = m_o + 1\})$*:* When $n\geq 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < m) \\ m \ (m\leq i\leq n-m) \\ n-i\sim m \ (n-m < i\leq n-1) \end{cases} \quad (47)$$

When $n < 2m$,

$$rank({}^1\boldsymbol{M}_i) = \begin{cases} i \ (1\leq i < n-m) \\ n-m \ (n-m\leq i\leq m) \\ n-i\sim n-m \ (m < i\leq n-1) \end{cases} \quad (48)$$

The proofs of these results are shown in subsection IV-D.

C. Mathematical Discriptions

1) Mathematical Definitions: $\boldsymbol{J}_n$ can be decomposed as

$$\boldsymbol{J}_n = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \quad (49)$$

and $\boldsymbol{J}_n^+$, the pseudo-inverse of $\boldsymbol{J}_n$, can be decomposed as

$$\boldsymbol{J}_n^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^T \quad (50)$$

In (49) and (50), $\boldsymbol{U}$ is $m\times m$ orthogonal matrix satisfying $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_m$, $\boldsymbol{V}$ is $n\times n$ orthogonal matrix satisfying $\boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}_n$, $\boldsymbol{\Sigma}$ is $m\times n$ matrix, which includes a diagonal matrix composing of non-zero singular values of $\boldsymbol{J}_n$ and the rest parts are all zero elements. Here, we will discuss the condition that $rank(\boldsymbol{J}_n) = m$. So, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^+$ can be denoted as

$$\boldsymbol{\Sigma} = \begin{array}{c} \\ m \end{array} \overset{\begin{array}{cc} m & \quad n-m \end{array}}{\begin{pmatrix} \sigma_1 & & \boldsymbol{0} & \\ & \ddots & & \boldsymbol{0} \\ \boldsymbol{0} & & \sigma_m & \end{pmatrix}} \quad (51)$$

and

$$\boldsymbol{\Sigma}^+ = \begin{array}{c} m \\ \\ n-m \end{array} \overset{m}{\begin{pmatrix} \sigma_1^{-1} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \sigma_m^{-1} \\ & \boldsymbol{0} & \end{pmatrix}} \quad (52)$$

In (51) and (52), $\sigma_1\geq\cdots\geq\sigma_m > 0$.

Generally, $\boldsymbol{V}$ can be defined with column vectors $\hat{\boldsymbol{v}}_i$ $(i = 1, 2, \cdots, n)$ as

$$\boldsymbol{V} = [\hat{\boldsymbol{v}}_1 \ \ \hat{\boldsymbol{v}}_2 \ \ \cdots \ \ \hat{\boldsymbol{v}}_n] \quad (53)$$

In (53), column vectors $\hat{\boldsymbol{v}}_j$ $(j = 1, \cdots, m)$ are obtained as

$$\boldsymbol{J}_n^T\boldsymbol{J}_n\hat{\boldsymbol{v}}_j = \hat{\boldsymbol{v}}_j\sigma_j^2 \quad (54)$$

and $\boldsymbol{V}$ can be redefined with row vectors $\check{\boldsymbol{v}}_i$ $(i = 1, 2, \cdots, n)$ as

$$\boldsymbol{V} = [\check{\boldsymbol{v}}_1, \check{\boldsymbol{v}}_2, \cdots, \check{\boldsymbol{v}}_n]^T \quad (55)$$

In addition, when $rank(\boldsymbol{J}_n) = m$, we know that $\boldsymbol{J}_n$ can be also decomposed as

$$\boldsymbol{J}_n = \boldsymbol{U}_m\boldsymbol{\Sigma}_m\boldsymbol{V}_m^T \quad (56)$$

and $\boldsymbol{J}_n^+$ can be decomposed as

$$\boldsymbol{J}_n^+ = \boldsymbol{V}_m \boldsymbol{\Sigma}_m^+ \boldsymbol{U}_m^T \tag{57}$$

In (56) and (57), $\boldsymbol{U}_m$ is $m \times m$ matrix satisfying $\boldsymbol{U}_m \boldsymbol{U}_m^T = \boldsymbol{U}_m^T \boldsymbol{U}_m = \boldsymbol{I}_m$, $\boldsymbol{U}_m$ and $\boldsymbol{U}$ are same. $\boldsymbol{V}_m^T$ is $m \times n$ matrix satisfying $\boldsymbol{V}_m^T \boldsymbol{V}_m = \boldsymbol{I}_m$, and $\boldsymbol{V}_m$ is defined using first m column vectors $\hat{\boldsymbol{v}}_j$ $(j = 1, 2, \cdots, m)$ in (53) as

$$\boldsymbol{V}_m = [\hat{\boldsymbol{v}}_1 \ \cdots \ \hat{\boldsymbol{v}}_m] \tag{58}$$

$\boldsymbol{V}_m$ is redefined referring to row vectors $\check{\boldsymbol{v}}_i$ $(i = 1, 2, \cdots, n)$ in (55) as

$$\boldsymbol{V}_m = [\check{\boldsymbol{v}}_{1,m}, \cdots, \check{\boldsymbol{v}}_{n,m}]^T \tag{59}$$

$\boldsymbol{V}_{n-m}$ is the rest block part of $\boldsymbol{V}$ except $\boldsymbol{V}_m$. So, $\boldsymbol{V}_{n-m}$ can be denoted using column vectors $\hat{\boldsymbol{v}}_j$ $(j = m+1, \cdots, n)$ in (53) as

$$\boldsymbol{V}_{n-m} = [\hat{\boldsymbol{v}}_{m+1} \ \cdots \ \hat{\boldsymbol{v}}_n] \tag{60}$$

$\boldsymbol{V}_{n-m}$ can be redenoted referring to row vectors $\check{\boldsymbol{v}}_i$ $(i = 1, 2, \cdots, n)$ in (55) as

$$\boldsymbol{V}_{n-m} = [\check{\boldsymbol{v}}_{1,(n-m)}, \cdots, \check{\boldsymbol{v}}_{n,(n-m)}]^T \tag{61}$$

$\boldsymbol{\Sigma}_m$ is $m \times m$ matrix, which is a diagonal matrix including m non-zero singular values of $\boldsymbol{J}_n$. $\boldsymbol{\Sigma}_m^+$ is also $m \times m$ diagonal matrix. So, $\boldsymbol{\Sigma}_m$ and $\boldsymbol{\Sigma}_m^+$ are denoted as

$$\boldsymbol{\Sigma}_m = \begin{array}{c} \\ m \end{array} \overset{m}{\begin{pmatrix} \sigma_1 & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \sigma_m \end{pmatrix}} \tag{62}$$

and

$$\boldsymbol{\Sigma}_m^+ = \begin{array}{c} \\ m \end{array} \overset{m}{\begin{pmatrix} \sigma_1^{-1} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \sigma_m^{-1} \end{pmatrix}} \tag{63}$$

And we divide $\boldsymbol{V}_m$ into two block matrices ($\boldsymbol{V}_{(n-m),m}$ and $\boldsymbol{V}_{m,m}$) and divide $\boldsymbol{V}_{n-m}$ into two block matrices ($\boldsymbol{V}_{(n-m),(n-m)}$ and $\boldsymbol{V}_{m,(n-m)}$), so that $\boldsymbol{V}$ can be redenoted as

$$\begin{aligned} \boldsymbol{V} &= [\boldsymbol{V}_m \ \ \boldsymbol{V}_{n-m}] \\ &= \begin{array}{c} n-m \\ m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{V}_{(n-m),m} & \boldsymbol{V}_{(n-m),(n-m)} \\ \boldsymbol{V}_{m,m} & \boldsymbol{V}_{m,(n-m)} \end{pmatrix}} \\ &= \begin{array}{c} n-m \\ m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{B} & \boldsymbol{D} \end{pmatrix}} \end{aligned} \tag{64}$$

2) Decomposition of $\boldsymbol{L}_n$: Firstly, we define

$$\boldsymbol{L}_n = \boldsymbol{I}_n - \boldsymbol{J}_n^+ \boldsymbol{J}_n \tag{65}$$

Then, from (34),

$${}^1\boldsymbol{M}_i = \boldsymbol{J}_i \boldsymbol{L}_n \tag{66}$$

If $rank(\boldsymbol{J}_n) = m$. Then, according to (49) and (50) and referring to (64), $\boldsymbol{L}_n$ can be decomposed as

$$\begin{aligned} \boldsymbol{L}_n &= \boldsymbol{I}_n - \boldsymbol{V}\boldsymbol{\Sigma}^+ \boldsymbol{U}^T \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^T \\ &= \boldsymbol{I}_n - \boldsymbol{V} \begin{array}{c} m \\ n-m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}} \boldsymbol{V}^T \\ &= \boldsymbol{V}\boldsymbol{V}^T - \boldsymbol{V} \begin{array}{c} m \\ n-m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}} \boldsymbol{V}^T \\ &= \boldsymbol{V} \begin{array}{c} m \\ n-m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-m} \end{pmatrix}} \boldsymbol{V}^T \\ &= n \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{V}_m & \boldsymbol{V}_{n-m} \end{pmatrix}} \begin{array}{c} m \\ n-m \end{array} \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-m} \end{pmatrix}} \boldsymbol{V}^T \\ &= n \overset{\begin{array}{cc} m & n-m \end{array}}{\begin{pmatrix} \boldsymbol{0} & \boldsymbol{V}_{n-m} \end{pmatrix}} \begin{array}{c} m \\ n-m \end{array} \overset{n}{\begin{pmatrix} \boldsymbol{V}_m^T \\ \boldsymbol{V}_{n-m}^T \end{pmatrix}} \\ &= n \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{n-m} \end{pmatrix}} \ n-m \overset{n}{\begin{pmatrix} \boldsymbol{V}_{n-m}^T \end{pmatrix}} \end{aligned} \tag{67}$$

In (67), because $rank(\boldsymbol{V}_{n-m}) = rank(\boldsymbol{V}_{n-m}^T) = n - m$, we can obtain

$$rank(\boldsymbol{L}_n) = n - m \tag{68}$$

D. Proofs of Results

We start these proofs by general relation of $rank({}^1\boldsymbol{M}_i)$ shown in (75) through decomposing ${}^1\boldsymbol{M}_i$. Here, firstly we divide $\boldsymbol{V}_{n-m}$ as

$$\boldsymbol{V}_{n-m} = \begin{array}{c} i \\ n-i \end{array} \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{i,(n-m)} \\ \boldsymbol{V}_{(n-i),(n-m)} \end{pmatrix}} \tag{69}$$

In (69), $\boldsymbol{V}_{i,(n-m)}$ is

$$\boldsymbol{V}_{i,(n-m)} = i \overset{n-m}{\begin{pmatrix} \check{\boldsymbol{v}}_{1,(n-m)} \\ \vdots \\ \check{\boldsymbol{v}}_{i,(n-m)} \end{pmatrix}} \tag{70}$$

and $\boldsymbol{V}_{(n-i),(n-m)}$ is

$$\boldsymbol{V}_{(n-i),(n-m)} = n-i \overset{n-m}{\begin{pmatrix} \check{\boldsymbol{v}}_{(i+1),(n-m)} \\ \vdots \\ \check{\boldsymbol{v}}_{n,(n-m)} \end{pmatrix}} \tag{71}$$

Then, according to (26), (67) and (69), ${}^1\boldsymbol{M}_i$ can be decomposed as

$$
\begin{aligned}
{}^1\boldsymbol{M}_i &= \boldsymbol{J}_i \boldsymbol{L}_n \\
&= \begin{array}{c} \\ m \end{array} \begin{array}{c} \begin{array}{cc} i & n-i \end{array} \\ \left(\tilde{\boldsymbol{J}}_i \quad \boldsymbol{0} \right) \end{array} \begin{array}{c} \\ n \end{array} \begin{array}{c} n-m \\ \left(\boldsymbol{V}_{n-m} \right) \end{array} \begin{array}{c} \\ n-m \end{array} \begin{array}{c} n \\ \left(\boldsymbol{V}_{n-m}^T \right) \end{array} \\
&= \begin{array}{c} \\ m \end{array} \begin{array}{c} i \\ \left(\tilde{\boldsymbol{J}}_i \right) \end{array} \begin{array}{c} \\ i \end{array} \begin{array}{c} n-m \\ \left(\boldsymbol{V}_{i,(n-m)} \right) \end{array} \begin{array}{c} \\ n-m \end{array} \begin{array}{c} n \\ \left(\boldsymbol{V}_{n-m}^T \right) \end{array} \quad (72)
\end{aligned}
$$

Then, we can obtain

$$
\begin{aligned}
rank({}^1\boldsymbol{M}_i) &= rank(\tilde{\boldsymbol{J}}_i \boldsymbol{V}_{i,(n-m)} \boldsymbol{V}_{n-m}^T) \\
&\geq rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)} \boldsymbol{V}_{n-m}^T) - i \\
&\geq rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) \\
&\quad + rank(\boldsymbol{V}_{n-m}^T) - i - (n-m) \\
&= rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) + (n-m) \\
&\quad - i - (n-m) \\
&= rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) - i \qquad (73)
\end{aligned}
$$

and

$$
\begin{aligned}
rank({}^1\boldsymbol{M}_i) &= rank(\tilde{\boldsymbol{J}}_i \boldsymbol{V}_{i,(n-m)} \boldsymbol{V}_{n-m}^T) \\
&\leq min\{rank(\tilde{\boldsymbol{J}}_i),\, rank(\boldsymbol{V}_{i,(n-m)}), \\
&\quad rank(\boldsymbol{V}_{n-m}^T)\} \\
&= min\{rank(\tilde{\boldsymbol{J}}_i),\, rank(\boldsymbol{V}_{i,(n-m)}), \\
&\quad n-m\} \qquad (74)
\end{aligned}
$$

According to (41) and (95) in “APPENDIX **??**” (the proof of (95) is shown in “APPENDICES A-**??**”), (73) and (74) can be denoted as

$$
\begin{aligned}
min\{i, m\} + min\{i, n-m\} - i \leq rank({}^1\boldsymbol{M}_i) \leq \\
min\{i, m, n-m\} \qquad (75)
\end{aligned}
$$

(1): When $\{n \geq 2m\} \cap \{1 \leq i < m\}$ or $\{n < 2m\} \cap \{1 \leq i < n-m\}$, by inputting these conditions into (75), we can obtain

$$rank({}^1\boldsymbol{M}_i) = i \qquad (76)$$

(2): When $\{n \geq 2m\} \cap \{m \leq i \leq n-m\}$, by inputting these conditions into (75), we can obtain

$$rank({}^1\boldsymbol{M}_i) = m \qquad (77)$$

(3): When $\{n < 2m\} \cap \{n-m \leq i \leq m\}$, by inputting these conditions into (75), we can obtain

$$rank({}^1\boldsymbol{M}_i) = n-m \qquad (78)$$

(4): When $\{n < 2m\} \cap \{m < i \leq n-1\}$, by inputting these conditions into (75), we can obtain

$$n - i \leq rank({}^1\boldsymbol{M}_i) \leq n-m \qquad (79)$$

(5): When $\{m = m_p \cup m = m_p + m_o\} \cap \{2 \leq m_p \leq 3\} \cap \{n \geq 2m\} \cap \{n-m < i \leq n-2\}$ or $\{m = m_p = 1 \cup m = m_o \cup m = m_o + 1\} \cap \{n \geq 2m\} \cap \{n-m < i \leq n-1\}$, by inputting these conditions into (75), we can obtain

$$n - i \leq rank({}^1\boldsymbol{M}_i) \leq m \qquad (80)$$

(6): When $\{m = m_p \cup m = m_p + m_o\} \cap \{2 \leq m_p \leq 3\} \cap \{n \geq 2m\} \cap \{i = n-1\}$, we can obtain

$${}^1\boldsymbol{M}_{n-1} = \tilde{\boldsymbol{J}}_{n-1} \boldsymbol{V}_{(n-1),(n-m)} \boldsymbol{V}_{n-m}^T \qquad (81)$$

By inputting (41) and (95) into (73), we can obtain

$$1 \leq rank({}^1\boldsymbol{M}_{n-1}) \qquad (82)$$

In addition, ${}^1\boldsymbol{M}_{n-1}$ can be rewritten as

$$
\begin{aligned}
{}^1\boldsymbol{M}_{n-1} &= \boldsymbol{J}_{n-1} \boldsymbol{L}_n \\
&= (\boldsymbol{J}_n - \Delta\boldsymbol{J}_n) \boldsymbol{L}_n = -\Delta\boldsymbol{J}_n \boldsymbol{L}_n \qquad (83)
\end{aligned}
$$

In (83), because $m \neq m_o$, $\Delta\boldsymbol{J}_n$ can be denoted as

$$\Delta\boldsymbol{J}_n = \Delta\boldsymbol{J}_{p,n} \qquad (84)$$

or according to (29), $\Delta\boldsymbol{J}_n$ is denoted as

$$\Delta\boldsymbol{J}_n = \begin{bmatrix} \Delta\boldsymbol{J}_{p,n} \\ \Delta\boldsymbol{J}_{o,n} \end{bmatrix} \qquad (85)$$

In (84) and (85), $\Delta\boldsymbol{J}_{p,n}$ is described as

$$\Delta\boldsymbol{J}_{p,n} = [\underbrace{{}^0\boldsymbol{z}_1 \times ({}^0\boldsymbol{R}_n {}^n\hat{\boldsymbol{p}}_E), \cdots, {}^0\boldsymbol{z}_n \times ({}^0\boldsymbol{R}_n {}^n\hat{\boldsymbol{p}}_E)}_{n}]\}m_p \qquad (86)$$

From (86), we know that all column vectors are the vertical vectors to ${}^0\boldsymbol{R}_n {}^n\hat{\boldsymbol{p}}_E$ in m_p-dimensional space. Therefore, these all column vectors in $\Delta\boldsymbol{J}_{p,n}$ can be thought that they are in $(m_p - 1)$-dimensional space. Then, we can obtain $rank(\Delta\boldsymbol{J}_{p,n}) \leq m_p - 1$. And because $rank(\Delta\boldsymbol{J}_{o,n}) \leq m_o$, we can obtain $rank(\Delta\boldsymbol{J}_n) \leq m_p - 1 + m_o = m - 1$. And because $rank(\boldsymbol{L}_n) = n - m \geq m - 1$ from (68), so, we can obtain

$$1 \leq rank({}^1\boldsymbol{M}_{n-1}) \leq m - 1 \qquad (87)$$

In this way, the results from (43) to (48) are proved in above six rough conditions as shown (76), (77), (78), (79), (80) and (87).

V. CONCLUSION

This work was supported by Grant-in-Aid for Scientific Research (C) 19560254. In this paper, based on the concept of avoidance manipulability, we present “Non-singular Configuration Assumption” for maximization of shape-changeable space expansion ($rank({}^1\boldsymbol{M}_i)$) of intermediate links, which is the most essential requirement for configuration optimization of manipulator with high avoidance manipulability. In the future, “Non-singular Configuration Assumption” will be used for an on-line control system of a redundant manipulator as the basic guarantee of high avoidance manipulability, where the system should be stopped once manipulator’s singular configuration is detected.

Appendix A
Proof of $rank(\boldsymbol{V}_{m,m}) = m$

According to (41), we can obtain $rank(\boldsymbol{J}_n) = m$, so, referring to (56), $\boldsymbol{J}_n$ can be decomposed as

$$\boldsymbol{J}_n = \boldsymbol{U}_m \boldsymbol{\Sigma}_m \boldsymbol{V}_m^T = \boldsymbol{R}_m \boldsymbol{V}_m^T \tag{88}$$

In (88), because $rank(\boldsymbol{U}_m) = m$ and $rank(\boldsymbol{\Sigma}_m) = m$, so $rank(\boldsymbol{R}_m) = rank(\boldsymbol{U}_m \boldsymbol{\Sigma}_m) = m$. Then, according to (88), we can obtain

$$\boldsymbol{V}_m^T = \boldsymbol{R}_m^{-1} \boldsymbol{J}_n \tag{89}$$

(89) can be rewritten as

$$[\boldsymbol{V}_{(n-m),m}^T,\ \boldsymbol{V}_{m,m}^T] = \boldsymbol{R}_m^{-1} \boldsymbol{J}_n \tag{90}$$

According to (90), we can obtain

$$\boldsymbol{V}_{m,m}^T = \boldsymbol{R}_m^{-1} \tilde{\boldsymbol{J}}_n^{n-m+1 \to n} \tag{91}$$

In (91), because $rank(\boldsymbol{R}_m^{-1}) = m$ and (41) ($rank(\tilde{\boldsymbol{J}}_n^{n-m+1 \to n}) = m$), we can obtain

$$rank(\boldsymbol{V}_{m,m}^T) = rank(\boldsymbol{V}_{m,m}) = m \tag{92}$$

Appendix B
$rank(\boldsymbol{V}_{i,(n-m)})$

When $1 \leq i < n-m$, $\boldsymbol{V}_{i,(n-m)}$ is one part of $\boldsymbol{V}_{(n-m),(n-m)}$ as

$$\boldsymbol{V}_{(n-m),(n-m)} = \begin{array}{c} \\ i \\ n-m-i \end{array} \begin{array}{c} n-m \\ \begin{pmatrix} \boldsymbol{V}_{i,(n-m)} \\ \boldsymbol{V}_{(n-m-i),(n-m)} \end{pmatrix} \end{array} \tag{93}$$

When $n-m \leq i \leq n$, $\boldsymbol{V}_{(n-m),(n-m)}$ is one part of $\boldsymbol{V}_{i,(n-m)}$ as

$$\boldsymbol{V}_{i,(n-m)} = \begin{array}{c} \\ n-m \\ i-n+m \end{array} \begin{array}{c} n-m \\ \begin{pmatrix} \boldsymbol{V}_{(n-m),(n-m)} \\ \boldsymbol{V}_{(i-n+m),(n-m)} \end{pmatrix} \end{array} \tag{94}$$

So, from (**??**),

$$rank(\boldsymbol{V}_{i,(n-m)}) = min\{i, n-m\} \tag{95}$$

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