Non-Singular Configuration Analyses of Redundant Manipulators for Optimizing Avoidance Manipulability

Tongxiao Zhang, Fufia Yu, Mamoru Minami, Osami Yasukura, Wei Song, Akira Yanou and Mingcong Deng
Mamoru Minami, Fufia Yu, Akira Yanou, Mingcong Deng are with Okayama University Tsushimanaka3-1-1 Okayama JAPAN, { minami, yufufia, yanou, deng} @suri.sys.okayama-u.ac.jp.
Osami Yasukura is with University of Fukui, Fukui Japan, yasukura@apphy.u-fukui.ac.jp.
Tongxiao Zhang is with China University of Mining and Technology, Jiangsu Xuzhou China, zhangtongxiaode@hotmail.com.
Wei Song is with University of Shanghai,Shanghai China, songwei5726@hotmail.com.

Abstract—This paper is concerned with a new concept of avoidance manipulability inspired from manipulability. The manipulability represents the ability to generate velocity at the tip of each link without any designated hand task. The avoidance manipulability represents the shape-changeability (avoidance ability) of each intermediate link when a prior hand task is given. The intermediate links represents all comprising links of robot except top link with end-effector. The avoidance matrices, $M_i$, $(i = 1, \ldots, n - 1)$ corresponding to all $i$-th intermediate links, are used for analyzing avoidance manipulability, resulting in that $\text{rank}(M_i)$ declares the shape-changeable space expansion and singular values of $M_i$ indicates the avoidance ability of $i$-th link. In this research, what assumption can guarantee mathematically the configurations with maximum $\text{rank}(M_i)$ is our main concern for maximizing shape-changeability to prepare effectively dynamic change of environment or sudden appearance of obstacles. Then we proved that our “Non-singular Configuration Assumption” can guarantee the maximum rank of $M_i$, through detailed decomposition analyses of $M_i$.

I. INTRODUCTION

Kinematically redundant manipulators have more DoF than necessary for accomplishing a given hand task. Nowadays, redundant manipulators are used for various kinds of tasks such as welding, sealing, grinding and contact tasks. Many kinematic researches are usually used to solve the problem of motion and obstacle avoidance of redundant manipulators discussing how to use the redundancy. Up to now, a variety of indices have been proposed for evaluation of the performance of robot manipulators. The manipulability ellipsoid [1] was presented to evaluate the static performance of a robot manipulator as an index evaluating the manipulator’s shape on the view point of how much the hand velocity can be generated by normalized joint velocity. Further, [2] formulated the relation of the redundancy and the priority order of multiple tasks. [3] proposed a control method of the redundancy based on priority order of tasks, and pointed out the effectiveness by actual experiments. A method that uses perturbation of a dumping element for avoiding obstacles along with singular configuration, and a regulation method of dumping element were discussed in [4]. The manipulability measurement was addressed for cooperative arms [5] and for dexterous hands [6] and was used in real-time control [7].

In this research, what assumption can guarantee mathematically the configurations with maximum $\text{rank}(M_i)$ is our main concern for maximizing shape-changeability to prepare effectively dynamic change of environment or sudden appearance of obstacles. Then we proved that our “Non-singular Configuration Assumption” can guarantee the maximum rank of $M_i$, through detailed decomposition analyses of $M_i$.

Fig. 1. Manipulability ellipsoids and Avoidance manipulability ellipsoids

addition, the manipulating force ellipsoid [8] was presented to evaluate the static torque-force transmission from the joints to the end-effector, while the dynamic manipulability ellipsoid [9] was presented as an index of the dynamic performance of a robot manipulator. Recent years, combining the dynamic manipulability ellipsoid with the manipulability force ellipsoid, the inertia matching ellipsoid [10] was proposed to characterize the dynamic torque-force transmission efficiency.

The researches mentioned above were an argument in a condition that an assumption guarantees the possibility that multiple avoiding motions could be realized. They did not consider how much residue redundant freedoms are remained at the links required to avoid obstacle. However, in an on-line system with dynamic environment, when a moving obstacle appears suddenly near the manipulator, it requires the manipulator to possess the ability to avoid this moving obstacle by changing its shape, which is so-called “Avoidance Manipulability”. In this background, as shown in Fig.1, we had presented the avoidance manipulability ellipsoid concept as an index evaluating shape-changeability of the manipulator [11], which is inspired from the manipulability concept [1]. In fact, the avoidance matrix $(M_i)$, which is important to analyze avoidance manipulability, had initially been defined and used for controlling the redundant manipulator’s configuration based
on prioritized multiple tasks [12]. However, the proposed controller can not decouple the interacting motions of multiple tasks even though the redundant degree be much higher than the required motion degree of the multiple tasks, stemming from incomplete definition of Jacobian matrix concerning the motion of what number of links the matrix describes. Contrastingly [12] with our definition of Jacobian matrix, the detailed difference and explanation are shown in sections III.

**II. REDUNDANT MANIPULATOR’S KINEMATICS**

**A. Analysis in Position Space**

Representing the position vector of i-th link by \( r_{p,i} \in R^{m_p} \) \((i = 1, 2, \cdots, n)\). \( m_p \) denotes the position dimension number of working space \((1 \leq m_p \leq 3)\), \( n \) denotes the number of the manipulator’s links and \( m_p < n \) because of redundancy. \( r_{p,i} \) is given as a function of \( q_i \) and defined as

\[
r_{p,i} = r_{p,i}(q_i) = [r_i(q_i), \cdots, r_{m_p,i}(q_i)]^T
\]

(1)

In (1), \( q_i \) with \( n \) elements is defined as

\[
q_i = [q_i, \cdots, q_i, 0, \cdots, 0]^T, \quad (i = 1, 2, \cdots, n)
\]

(2)

In addition, according to Fig.3, \( r_{p,i}(q_i) \) can be denoted as

\[
r_{p,i}(q_i) = \sum_{j=1}^{i} \Delta r_{p,j}(q_j)
\]

(3)

By differentiating \( r_{p,i}(q_i) \) in (3) with time, we can obtain

\[
\dot{r}_{p,i}(q_i) = \frac{\partial r_{p,i}(q_i)}{\partial q_i} \dot{q}_i + \cdots + \frac{\partial r_{p,i}(q_i)}{\partial q_i} \dot{q}_i = J_{p,i} \dot{q}_i
\]

(4)

Then, we can obtain the position Jacobian matrix \( J_{p,i} \) \((i = 1, 2, \cdots, n)\) in (4) as follows:

\[
J_{p,i} = \left[ \frac{\partial \Delta r_{p,1}(q_1)}{\partial q_1}, \cdots, \frac{\partial \Delta r_{p,i}(q_i)}{\partial q_i}, \frac{\partial \Delta r_{p,i+1}(q_{i+1})}{\partial q_i}, \cdots, \frac{\partial \Delta r_{p,n}(q_n)}{\partial q_i} \right]_{m_p}^n
\]

(5)

If we redefine \( \Delta J_{p,j} \) as

\[
\Delta J_{p,j} = \left[ \frac{\partial \Delta r_{p,1}(q_1)}{\partial q_j}, \cdots, \frac{\partial \Delta r_{p,j}(q_j)}{\partial q_j}, \frac{\partial \Delta r_{p,j+1}(q_{j+1})}{\partial q_j}, \cdots, \frac{\partial \Delta r_{p,n}(q_n)}{\partial q_j} \right]_{m_p}^n
\]

(6)

\( J_{p,i} \) \((i = 1, 2, \cdots, n)\) can be denoted as:

\[
J_{p,i} = \sum_{j=1}^{i} \Delta J_{p,j}
\]

(7)

In this way, \( J_{p,n} \) can be denoted as

\[
J_{p,n} = \sum_{j=1}^{n} \Delta J_{p,j} = J_{p,i} + \sum_{j=i+1}^{n} \Delta J_{p,j}
\]

(8)

In addition, referring to Fig.4, we know

\[
0_{p,i+1,k} = \sum_{j=k}^{i} \Delta r_{p,j}(q_j) = \sum_{j=k}^{i} 0_{R_j} \hat{p}_{j+1}
\]

(9)

In (9), \( 0_{R_j} \) is rotation matrix denoting the relation between \( \Sigma_0 \) and \( \Sigma_j \), \( \hat{p}_{j+1} \) is the constant denoting position vector from the origin of \( \Sigma_j \) to the one of \( \Sigma_{j+1} \) with respect to \( \Sigma_j \). Then, we can obtain

\[
\Delta r_{p,j}(q_j) = 0_{R_j} \hat{p}_{j+1}
\]

(10)

Then, by differentiating \( \Delta r_{p,j}(q_j) \) with time, we can obtain

\[
\frac{d \Delta r_{p,j}(q_j)}{dt} = \frac{\partial \Delta r_{p,j}(q_j)}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial \Delta r_{p,j}(q_j)}{\partial q_j} \dot{q}_j
\]

(11)
If \( q \) since, joint defined as
\[
0 = \text{(6)}, \quad \text{we can obtain}
\]
coordinates represent rotational axes. Substituting (12) into
joints in (14),
\[
\frac{d}{dt} (R_i R_j) = R_i R_j \frac{d}{dt} R_j + R_j \frac{d}{dt} R_i
\]
where, \( R_i \) denotes the unit vector of axis direction of \( i \)-th joint defined as
\[
R_i = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}
\]
and in (14), \( e_z = [0, 0, 1]^T \) providing z-axes of all link coordinates represent rotational axes. Substituting (12) into
(6), we can obtain
\[
J_{p,i} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}
\]
If the \( k \)-th joint is prismatic, the translational direction is represented by \( z_k \), i.e., then \( J_{p,i} \) is defined as
\[
J_{p,i} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}
\]
B. Analysis in Orientation Space

Representing the orientation vector of each link by \( r_{o,i} \in \mathbb{R}^{m_o} \). Here, \( m_o \) denotes the orientation dimension number of working space \((1 \leq m_o \leq 3)\). If \( r_{o,i} \) is represented by a rather common definition of “Euler angles” \( \phi_i, \theta_i, \psi_i \), it is given as a function of \( q \) and defined as
\[
r_{o,i}(q_i) = [\phi_i(q_i), \theta_i(q_i), \psi_i(q_i)]^T
\]
By differentiating \( r_{o,i} \) in (18) with time, we can obtain
\[
\dot{r}_{o,i}(q_i) = \frac{\partial r_{o,i}(q_i)}{\partial q_n} \dot{q}_n = J_{o,i} \dot{q}_n
\]
In addition, the relation between angular velocity vector \( \omega_i \) and \( \dot{r}_{o,i}(q_i) \) is
\[
\omega_i = \begin{bmatrix} -\sin\phi_i & \cos\phi_i & 0 \\ -\cos\phi_i & -\sin\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{q}_n
\]
In (20), providing z-axes of all links represent rotational axes, \( J_{o,i} \) is denoted as
\[
J_{o,i} = \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix}
m_o
\]
for \( m_o \leq 3 \) orientation space. Then, we can define a general orientation dimension number of \( o \), \( o,i \), \( R \), and \( \theta \), \( \psi \), \( \phi \) to be denoted as
\[
J_{o,i} = \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix}
m_o
\]
Using similar (7), \( J_{o,i} \) can be denoted as
\[
J_{o,i} = \sum_{i=1}^{n} \Delta J_{o,i}
\]
C. Analysis in Both Position and Orientation Spaces

According to above analyses of Jacobian matrices in position space \((1 \leq m_p \leq 3)\) and orientation space \((1 \leq m_o \leq 3)\) respectively. Firstly, when \( m_p = 3 \), \( r_{p,i}(q_i) = [x(q_i), y(q_i), z(q_i)]^T \). Then, from (5) we can define a general position Jacobian matrix \((3 \times n)\) as
\[
J_{p,i} = \begin{bmatrix} \frac{\partial x(q_i)}{\partial q_n} & \cdots & \frac{\partial x(q_i)}{\partial n} & 0 \\ \frac{\partial y(q_i)}{\partial q_n} & \cdots & \frac{\partial y(q_i)}{\partial n} & 0 \\ \frac{\partial z(q_i)}{\partial q_n} & \cdots & \frac{\partial z(q_i)}{\partial n} & 0 \end{bmatrix}
\]
Next, when \( m_o = 3 \) and \( r_{o,i}(q_i) \) is defined as (18) in orientation space. Then, we can define a general orientation Jacobian matrix \((3 \times n)\) as
\[
J_{o,i} = \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix}
\]
Finally, if a general Jacobian matrix \((6 \times n)\) in the maximum space of \( m = m_p + m_o = 6 \) such as \( r_{i}(q_i) = [x(q_i), y(q_i), z(q_i), \phi_i(q_i), \theta_i(q_i), \psi_i(q_i)]^T \) is defined as
\[
J_{i} = \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix}
\]
In this way, according to $J_b^{m=6}$, we can define any Jacobian matrix in any kind of space as

$$J_i = U^m J_b^{m=6} = \begin{bmatrix} \hat{J}_{i,1} & \cdots & \hat{J}_{i,n-1} \end{bmatrix} m = [\hat{J}_i, 0]$$  \hspace{1cm} (26)

In (26) providing $i = n$, $U^m$ is a $m \times 6$ matrix and is used to select objective space used selectively for hand task. For example, when the objective space is given by 3-dimensional such as $r_i(q_i) = [x(q_i), y(q_i), \phi_i(q_i)]^T$, $U^m$ is $3 \times 6$ matrix as

$$U^m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 
\end{bmatrix}$$  \hspace{1cm} (27)

In addition, being similar with (7) and (23), we can define

$$J_i = \sum_{j=1}^{i} \Delta J_j$$  \hspace{1cm} (28)

and

$$\Delta J_j = \begin{bmatrix} \Delta J_{p,j} \\ \Delta J_{o,j} \end{bmatrix}$$  \hspace{1cm} (29)

### III. AVOIDANCE MANIPULABILITY

Here we assume that the desired trajectory $(r_{nd})$ and the desired velocity of the manipulator’s hand $(\dot{r}_{nd})$ are given as primary task. Then, according to (30),

$$\dot{r}_{nd} = J_n \dot{q}_n$$  \hspace{1cm} (30)

we can obtain

$$\dot{q}_n = J_n^+ \dot{r}_{nd} + (I_n - J_n^+ J_n)^{-1} l$$  \hspace{1cm} (31)

In (31), $J_n$ is Jacobian matrix differentiated $r_n$ by $q_n$ ($\dot{r}_n = J_n \dot{q}_n$), $J_n^+$ is pseudo-inverse of $J_n$, $I_n$ is $n \times n$ unit matrix, and $l$ is an arbitrary vector satisfying $l \in R^m$. The left superscript “1” of $l$ means the first avoidance sub-task executed by using redundant DoF. The following definitions about the left superscript “1” are also. By substituting (31) into $\dot{r}_{di} = J_i \dot{q}_n$, the relation of $\dot{r}_{di}$ and $\dot{r}_{nd}$ is denoted as

$$\dot{r}_{di} = J_i J_n^+ \dot{r}_{nd} + J_i (I_n - J_n^+ J_n)^{-1} l$$  \hspace{1cm} (32)

Then, we define two variables shown as

$$\Delta^1 \dot{r}_{di} = \dot{r}_{di} - J_i J_n^+ \dot{r}_{nd}$$  \hspace{1cm} (33)

and

$$1 M_i \Delta^1 \dot{r}_{di} = J_i (I_n - J_n^+ J_n)$$  \hspace{1cm} (34)

In (33), $\Delta^1 \dot{r}_{di}$ is called by “the first avoidance velocity”. In (34), $1 M_i$ is $m \times n$ matrix called by “the first avoidance matrix”. Then, $\Delta^1 \dot{r}_{di}$ can be rewritten as

$$\Delta^1 \dot{r}_{di} = 1 M_i^+ l$$  \hspace{1cm} (35)

The relation between $\dot{r}_{di}$ and $\Delta^1 \dot{r}_{di}$ is shown in Fig.5. From (35), we can obtain $l$ shown as

$$l = 1 M_i^+ \Delta^1 \dot{r}_{di} + (I_n - 1 M_i^+ 1 M_i)^{-1} 1$$  \hspace{1cm} (36)

In (36), $1 M_i^+$ is pseudo-inverse of $1 M_i$ and $2 l$ is an arbitrary vector satisfying $2 l \in R^m$. From (36), we can obtain

$$\|l\|^2 \geq \Delta^1 \dot{r}_{di}^T (1 M_i^+)^T 1 M_i^+ \Delta^1 \dot{r}_{di}$$  \hspace{1cm} (37)

Assuming that $l$ is restricted as $\|l\| \leq 1$, then the extent where $\Delta^1 \dot{r}_{di}$ can move is denoted as

$$\Delta^1 \dot{r}_{di}^T (1 M_i^+)^T 1 M_i^+ \Delta^1 \dot{r}_{di} \leq 1$$  \hspace{1cm} (38)

When $\text{rank}(1 M_i) = m$, (38) represents that the first avoidance velocity $\Delta^1 \dot{r}_{di}$ can be described by an ellipsoid expanded in $m$-dimensional space, which indicates $\Delta^1 \dot{r}_{di}$ can be freely realized in $m$-dimensional task space. The ellipsoid represented by (38) is named as the first complete avoidance manipulability ellipsoid. However, when $\text{rank}(1 M_i) = p < m$, the extent of the new first avoidance velocity $\Delta^1 \dot{r}_{di}$ is denoted as

$$\Delta^1 \dot{r}_{di}^T (1 M_i^+)^T 1 M_i^+ \Delta^1 \dot{r}_{di} \leq 1$$  \hspace{1cm} (39)

This new first avoidance velocity $\Delta^1 \dot{r}_{di}$ can be described by an ellipsoid expanded in $p$-dimensional space. The ellipsoid represented by (39) is named as the first partial avoidance manipulability ellipsoid. Because $p < m$, the first partial avoidance manipulability ellipsoid can be thought as a segment of the first complete avoidance manipulability ellipsoid as first and third links shown in Fig.2(a). Thus $\text{rank}(1 M_i)$ determines the possible avoidance dimension of $i$-th link, therefore the condition to give $\text{rank}(1 M_i)$ maximum number is essential for configuration control and avoidance control to maximize the shape-changeability degree. Next we will propose an assumption and prove it guarantees the rank of $1 M_i$.

### IV. ANALYSIS of $\text{rank}(1 M_i)$

#### A. Non-singular Configuration Assumption

“Non-singular Configuration Assumption” is

$$\text{rank}(J_i^{\nu-\nu+m-1}) = \min\{i, m\} \ (1 \leq \nu \leq i - m + 1)$$  \hspace{1cm} (40)

In (40), $J_i^{\nu-\nu+m-1}$ indicates the matrices including the $m$ column vectors sequentially chosen from the first $i$ columns
of $J_i$ without the last $n - i$ zero columns. That is the $m$ column vectors sequentially chosen from $J_i$ as

$$rank(\tilde{J}_i^{\nu-n+m-1}) = \min\{i, m\} \quad (1 \leq \nu \leq n - m + 1) \quad (41)$$

In (41), for example, when $i = n$ and $\nu = n - m + 1$,

$$\tilde{J}_n^{n-m+1-n} = [\tilde{J}_{n,n-m+1}, \cdots, \tilde{J}_{n,n}] \quad (42)$$

Non-singular Configuration Assumption is a kind of mathematical denotation, which corresponds to the non-singular configuration described as Fig.6 in robot field.

B. Results

By (41), we will prove that we can obtain “Results” of $rank(1^iM_i)$ ($i = 1, 2, \cdots, n - 1$) as follows:

1) Results in Both Position and Orientation Spaces ($\{m = m_p + m_o\} \cap \{2 \leq m_p \leq 3\}$): When $n \geq 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < m) \\ m (m \leq n - m) \\ n - i \cdot m (n - m < i \leq n - 2) \\ 1 \sim m - 1 (i = n - 1) \end{cases} \quad (43)$$

When $n < 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < n - m) \\ n - m (n - m \leq i \leq m) \\ n - i \cdot n - m (m < i \leq n - 1) \end{cases} \quad (44)$$

2) Results in Position Space ($\{m = m_p\} \cap \{2 \leq m_p \leq 3\}$): When $n \geq 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < m) \\ m (m \leq n - m) \\ n - i \cdot m (n - m < i \leq n - 2) \\ 1 \sim m - 1 (i = n - 1) \end{cases} \quad (45)$$

When $n < 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < n - m) \\ n - m (n - m \leq i \leq m) \\ n - i \cdot n - m (m < i \leq n - 1) \end{cases} \quad (46)$$

3) Other Results ($\{m = m_p + 1\} \cup \{m = m_o + 1\}$): When $n \geq 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < m) \\ m (m \leq n - m) \\ n - i \cdot m (n - m < i \leq n - 1) \end{cases} \quad (47)$$

When $n < 2m$,

$$rank(1^iM_i) = \begin{cases} i (1 \leq i < n - m) \\ n - m (n - m \leq i \leq m) \\ n - i \cdot n - m (m < i \leq n - 1) \end{cases} \quad (48)$$

The proofs of these results are shown in subsection IV-D.

C. Mathematical Discriptions

1) Mathematical Definitions: $J_n$ can be decomposed as

$$J_n = U \Sigma V^T \quad (49)$$

and $J_n^+$, the pseudo-inverse of $J_n$, can be decomposed as

$$J_n^+ = V \Sigma^+ U^T \quad (50)$$

In (49) and (50), $U$ is $m \times m$ orthogonal matrix satisfying $UU^T = U^T U = I_m$, $V$ is $n \times n$ orthogonal matrix satisfying $VV^T = V^T V = I_n$, $\Sigma$ is $m \times n$ matrix, which includes a diagonal matrix composing of non-zero singular values of $J_n$ and the rest parts are all zero elements. Here, we will discuss the condition that $rank(J_n) = m$. So, $\Sigma$ and $\Sigma^+$ can be denoted as

$$\Sigma = m \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \cdots & \sigma_m \end{pmatrix} \quad (51)$$

and

$$\Sigma^+ = m \begin{pmatrix} \sigma_1^{-1} & \cdots & 0 \\ 0 & \cdots & \sigma_m^{-1} \end{pmatrix} \quad (52)$$

In (51) and (52), $\sigma_1 \geq \cdots \geq \sigma_m > 0$.

Generally, $V$ can be defined with column vectors $\hat{v}_i (i = 1, 2, \cdots, n)$ as

$$V = [\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n] \quad (53)$$

In (53), column vectors $\hat{v}_j (j = 1, \cdots, m)$ are obtained as

$$J_n^T J_n \hat{v}_j = \hat{v}_j \sigma_j^2 \quad (54)$$

and $V$ can be redefined with row vectors $\hat{v}_i (i = 1, 2, \cdots, n)$ as

$$V = [\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n]^T \quad (55)$$

In addition, when $rank(J_n) = m$, we know that $J_n$ can be also decomposed as

$$J_n = U_m \Sigma_n V_m^T \quad (56)$$
and \( J_n^+ \) can be decomposed as
\[
J_n^+ = V_m \Sigma_m^+ U_m^T \tag{57}
\]
In (56) and (57), \( U_m \) is a \( m \times m \) matrix satisfying \( U_m U_m^T = U_m^T U_m = I_m \) and \( U \) are same. \( V_m^+ \) is a \( m \times n \) matrix satisfying \( V_m^+ V_m = I_m \), and \( V_m \) is defined using first \( m \) column vectors \( \hat{\nu}_j \) (\( j = 1, 2, \ldots, m \)) in (53) as
\[
V_m = [\hat{\nu}_1 \cdots \hat{\nu}_m] \tag{58}
\]
\( V_m \) is redefined referring to row vectors \( \tilde{\nu}_i \) (\( i = 1, 2, \ldots, n \)) in (55) as
\[
V_{n-m} = [\tilde{\nu}_{m+1} \cdots \tilde{\nu}_n] \tag{60}
\]
\( V_{n-m} \) can be redefined referring to row vectors \( \tilde{\nu}_i \) (\( i = 1, 2, \ldots, n \)) in (55) as
\[
V_{n-m} = [\tilde{\nu}_{1,(n-m)} \cdots \tilde{\nu}_{n,(n-m)}]^T \tag{61}
\]
\( \Sigma_m \) is an \( m \times m \) matrix, which is a diagonal matrix including \( m \) non-zero singular values of \( J_n \). \( \Sigma_m^+ \) is also \( m \times m \) diagonal matrix. So, \( \Sigma_m \) and \( \Sigma_m^+ \) are denoted as
\[
\Sigma_m = m \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ & \ddots & & \\ & & \sigma_m \end{pmatrix} \tag{62}
\]
and
\[
\Sigma_m^+ = m \begin{pmatrix} \sigma_1^{-1} & 0 & \cdots & 0 \\ & \ddots & & \\ & & \sigma_m^{-1} \end{pmatrix} \tag{63}
\]
And we divide \( V_m \) into two block matrices \( (V_{(n-m),m}, V_{m,(n-m)}) \) and divide \( V_{n-m} \) into two block matrices \( (V_{(n-m),(n-m)}, V_{m,(n-m)}) \), so that \( V \) can be redefined as
\[
V = \begin{bmatrix} V_m & V_{n-m} \end{bmatrix} \tag{64}
\]
2) **Decomposition of \( L_n \):** Firstly, we define
\[
L_n = I_n - J_n^+ J_n \tag{65}
\]
Then, from (34),
\[
1 M_i = J_i L_n \tag{66}
\]
If \( \text{rank}(J_n) = m \). Then, according to (49) and (50) and referring to (64), \( L_n \) can be decomposed as
\[
L_n = I_n - \Sigma_n U^T U \Sigma_n V^T \tag{67}
\]
In (67), because \( \text{rank}(V_{n-m}) = \text{rank}(V_{n-m}^T) = n - m \), we can obtain
\[
\text{rank}(L_n) = n - m \tag{68}
\]
D. **Proofs of Results**

We start these proofs by general relation of \( \text{rank}(1 M_i) \) shown in (75) through decomposing \( 1 M_i \). Here, firstly we divide \( V_{n-m} \) as
\[
V_{n-m} = i \begin{pmatrix} V_{i,(n-m)} \\ V_{(n-i),(n-m)} \end{pmatrix} \tag{69}
\]
In (69), \( V_{i,(n-m)} \) is
\[
V_{i,(n-m)} = \begin{bmatrix} \tilde{\nu}_{1,(n-m)} \\ \vdots \\ \tilde{\nu}_{n,(n-m)} \end{bmatrix} \tag{70}
\]
and \( V_{(n-i),(n-m)} \) is
\[
V_{(n-i),(n-m)} = [\begin{array}{c} \tilde{\nu}_{(i+1),(n-m)} \\ \vdots \\ \tilde{\nu}_{n,(n-m)} \end{array}] \tag{71}
\]
Then, according to (26), (67) and (69), \( \text{rank}(1M_i) \) can be decomposed as
\[
1M_i = J_iL_n = m \begin{pmatrix} i & n-i \\ \bar{J}_i & 0 \end{pmatrix} n \begin{pmatrix} V_{n-m} \\ V^T_{n-m} \end{pmatrix} n = m \begin{pmatrix} \bar{J}_i \\ V_{i,(n-m)} \end{pmatrix} n = m \begin{pmatrix} \bar{J}_i \\ V_{i,(n-m)} \end{pmatrix} n-m \begin{pmatrix} \bar{J}_i \\ V^T_{n-m} \end{pmatrix} n. \tag{72}
\]

Then, we can obtain
\[
\text{rank}(1M_i) = \text{rank}(\bar{J}_iV_{i,(n-m)}V^T_{n-m}) \geq \text{rank}(\bar{J}_i) + \text{rank}(V_{i,(n-m)}V^T_{n-m}) - i \geq \text{rank}(\bar{J}_i) + \text{rank}(V_{i,(n-m)}) + (n-m) - i - (n-m) = \text{rank}(\bar{J}_i) + \text{rank}(V_{i,(n-m)}) - i \tag{73}
\]

and
\[
\text{rank}(1M_i) = \min\{\text{rank}(\bar{J}_i), \text{rank}(V_{i,(n-m)})\} \leq \text{rank}(\bar{J}_i) + \text{rank}(V_{i,(n-m)}) - i \leq m - n \tag{74}
\]

According to (41) and (95) in “APPENDIX ??” (the proof of (95) is shown in “APPENDICES A-??”), (73) and (74) can be denoted as
\[
\min\{i, m\} + \min\{i, n - m\} - i \leq \text{rank}(1M_i) \leq \min\{i, m, m - n\} \tag{75}
\]

(1): When \( n \geq 2m \cap \{1 \leq i < m\} \) or \( n < 2m \cap \{1 \leq i < n - m\} \), by inputting these conditions into (75), we can obtain
\[
\text{rank}(1M_i) = \text{rank}(\bar{J}_i) = i \tag{76}
\]

(2): When \( n \geq 2m \cap \{m \leq i \leq n - m\} \), by inputting these conditions into (75), we can obtain
\[
\text{rank}(1M_i) = m \tag{77}
\]

(3): When \( n < 2m \cap \{n - m \leq i \leq m\} \), by inputting these conditions into (75), we can obtain
\[
\text{rank}(1M_i) = n - m \tag{78}
\]

(4): When \( n < 2m \cap \{m < i \leq n - 1\} \), by inputting these conditions into (75), we can obtain
\[
n - i \leq \text{rank}(1M_i) \leq n - m \tag{79}
\]

(5): When \( m = m_p \cup m = m_p + m_o \cap \{2 \leq m_p \leq 2\cap \{n \geq 2m\} \cap \{n - m < i \leq n - 2\} \) or \( m = m_p + 1 \cup m = m_p + m_o + 1 \cap \{n \geq 2m\} \cap \{n - m < i \leq n - 1\} \), by inputting these conditions into (75), we can obtain
\[
n - i \leq \text{rank}(1M_i) \leq m \tag{80}
\]

By inputting (41) and (95) into (73), we can obtain
\[
1M_{n-1} = \bar{J}_{n-1} V_{(n-1),(n-m)}V^T_{n-m} \tag{81}
\]

Then, we can obtain
\[
1 \leq \text{rank}(1M_{n-1}) \tag{82}
\]

In addition, \( 1M_{n-1} \) can be rewritten as
\[
1M_{n-1} = J_{n-1}L_n = (J_n - \Delta J_n) L_n = -\Delta J_nL_n \tag{83}
\]

In (83), because \( m \neq m_o \), \( \Delta J_n \) can be denoted as
\[
\Delta J_n = \Delta J_{p,n} \tag{84}
\]

or according to (29), \( \Delta J_n \) is denoted as
\[
\Delta J_n = \begin{bmatrix} \Delta J_{p,n} \\ \Delta J_{o,n} \end{bmatrix} \tag{85}
\]

In (84) and (85), \( \Delta J_{p,n} \), is described as
\[
\Delta J_{p,n} = \begin{bmatrix} \tilde{z}_1 \times (0 R^n E_1) \cdots \tilde{z}_n \times (0 R^n E_n) \end{bmatrix} m_p \tag{86}
\]

From (86), we know that all column vectors are the vertical vectors to \( 0 R^n E \) in \( m_p \)-dimensional space. Therefore, these all column vectors in \( \Delta J_{p,n} \) can be thought that they are in \( (m_p - 1) \)-dimensional space. Then, we can obtain
\[
\text{rank}(\Delta J_{p,n}) \leq m_p - 1. \tag{87}
\]

And because \( \text{rank}(\Delta J_{o,n}) \leq m_o = m - 1 \), we can obtain
\[
\text{rank}(\Delta J_n) \leq m_p - 1 + m_o = m - 1. \tag{88}
\]

Therefore, \( \text{rank}(\Delta J_n) \leq m - 1 \) from (86), so we can obtain
\[
1 \leq \text{rank}(1M_{n-1}) \leq m - 1 \tag{89}
\]

In this way, the results from (43) to (48) are proved in above six rough conditions as shown (76), (77), (78), (79), (80) and (87).

V. CONCLUSION

This work was supported by Grant-in-Aid for Scientific Research (C) 19560254. In this paper, based on the concept of avoidance manipulability, we present “Non-singular Configuration Assumption” for maximization of shape-changeable space expansion (rank(1M_i)) of intermediate links, which is the most essential requirement for configuration optimization of manipulator with high avoidance manipulability. In the future, “Non-singular Configuration Assumption” will be used for an on-line control system of a redundant manipulator as the basic guarantee of high avoidance manipulability, where the system should be stopped once manipulator’s singular configuration is detected.
APPENDIX A

PROOF of rank($V_{m,m}$) = m

According to (41), we can obtain rank($J_n$) = m, so, referring to (56), $J_n$ can be decomposed as

$$J_n = U_n \Sigma_m V_m^T = R_m V_m^T$$  (88)

In (88), because rank($U_m$) = m and rank($\Sigma_m$) = m, so rank($R_m$) = rank($U_m \Sigma_m$) = m. Then, according to (88), we can obtain

$$V_m^T = R_m^{-1} J_n$$  (89)

(89) can be rewritten as

$$[V_{(n-m),m}^T, V_{m,m}^T] = R_m^{-1} J_n$$  (90)

According to (90), we can obtain

$$V_{m,m}^T = R_m^{-1} J_n$$  (91)

In (91), because rank($R_m^{-1}$) = m and (41), (rank($J_n^{m-m+1-n}$) = m), we can obtain

$$\text{rank}(V_{m,m}^T) = \text{rank}(V_{m,m}) = m$$  (92)

APPENDIX B

rank($V_{i,(n-m)}$)

When $1 \leq i < n - m$, $V_{i,(n-m)}$ is one part of $V_{(n-m),(n-m)}$ as

$$V_{(n-m),(n-m)} = \begin{pmatrix} V_{i,(n-m)} \\ n - m - i \end{pmatrix}$$  (93)

When $n - m \leq i \leq n$, $V_{(n-m),(n-m)}$ is one part of $V_{i,(n-m)}$ as

$$V_{i,(n-m)} = \begin{pmatrix} V_{(n-m),(n-m)} \\ n - m \end{pmatrix}$$  (94)

So, from (??),

$$\text{rank}(V_{i,(n-m)}) = \text{min}\{i, n - m\}$$  (95)

REFERENCES